

# Non-singular inflationary universe from polymer matter

Golam Mortuza Hossain,<sup>\*</sup> Viqar Husain,<sup>†</sup> and Sanjeev S. Seahra<sup>‡</sup>

*Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3*

(Dated: October 19, 2009)

We consider a polymer quantization of a free massless scalar field in a homogeneous and isotropic cosmological spacetime. This quantization method assumes that field translations are fundamentally discrete, and is related to but distinct from that used in loop quantum gravity. The semi-classical Friedman equation yields a universe that is non-singular and non-bouncing, without quantum gravity. The model has an early de Sitter-like inflationary phase with sufficient expansion to resolve the horizon and entropy problems, and a built in mechanism for a graceful exit from inflation.

**Motivation** The standard model of cosmology is in remarkable agreement with current observations. The model relies on the existence of a sustained period of accelerated expansion in the early universe. This inflationary epoch solves the flatness, horizon, and entropy problems, and as a bonus, provides a mechanism for the generation of primordial perturbations. However a drawback is that this epoch is put in by hand.

An important goal for fundamental theory therefore is to provide a solid theoretical foundation for inflation. This has led to much effort especially in string theory, although a conclusive and widely accepted answer is not currently available. An alternative approach known as loop quantum gravity (LQG), has also been much studied, especially in its application to cosmology, (known as LQC) [1, 2]. One of the main results in this area is that the big bang singularity of general relativity (GR) is not present in this model [3]. In LQC there is a quantum bounce [4] that occurs when the density of matter reaches a critical value  $\rho_c \sim M_{\text{pl}}^4$ , where  $M_{\text{pl}}$  is the Planck mass [5]. The bounce is followed by a short inflationary phase, but the amount of accelerated expansion is not enough to address phenomenological questions [6, 7]. Thus, so far no natural mechanism for sufficient inflation has emerged in this approach.

Both LQG and its mini-superspace realization to cosmology are formulated using a background independent quantization procedure (“polymer quantization”), where the inner product used is independent of the manifold metric. This is an important ingredient since the metric is a dynamical field in quantum gravity. All the work in LQC relies on applying this quantization to the geometric variables, but the usual Schrödinger quantization is used for the matter degrees of freedom, which is typically a homogeneous minimally coupled scalar field.

Polymer quantization may be viewed as a separate development in its own right, and is applicable to any classical theory whether or not it contains gravity [8, 9, 10, 11, 12]. Its central feature is that momentum operators are not realized directly as in Schrödinger quantum mechanics because of a built in notion of discreteness, but arise indirectly through translation operators. (There is a “dual” representation where it is the configuration variables that are represented only in an

exponentiated form [9].)

In this Letter we show that singularity resolution, sufficient inflation, and a graceful exit are all natural consequences of classical gravity sourced by polymer quantized matter. This approach may be viewed as the “polymer semiclassical approximation.” Unlike LQC, where singularity avoidance results from polymer geometry, all our results follow from solely from polymer matter coupled to classical geometry.

The basic insight from which these results follow is that polymer quantization of matter puts an upper bound on its kinetic energy, which naturally bounds curvature through the Hamiltonian constraint of GR. How this is realized in detail is described by using the massless scalar field coupled to the Friedmann-Robertson-Walker (FRW) geometry, the same model that was studied in LQC [5]. However, the procedure and variables we use for quantization are quite different from those of LQC – the only common feature is the use of polymer quantization.

We assume that the scale associated with this quantization is given by a mass parameter  $M_\star$  (which is a priori distinct from the Planck mass  $M_{\text{pl}}$ ). We show in this quantization that the matter energy density is bounded:  $\rho \lesssim M_\star^4$ . We also show that the universe approaches a de Sitter phase in the past for generic choices of parameters; this nonsingular inflationary universe replaces the bounce found in some models (such as LQC). Furthermore, for  $M_\star^4 \ll M_{\text{pl}}^4$  the curvature of the universe remains much less than the Planck scale, implying the quantum gravity effects are negligible and the classical treatment of geometry is justified. There is a special limiting case where the universe approaches Minkowski space in the asymptotic past.

**Polymer quantization** The model we consider is the massless scalar field minimally coupled to gravity. The total Hamiltonian of this system is the constraint  $H_g + H_\phi = 0$ , where  $H_g$  and  $H_\phi$  are the standard Hamiltonians for the gravity and matter sectors, respectively. The matter Hamiltonian is

$$H_\phi = \int d^3x N \left[ \frac{1}{2\sqrt{q}} P_\phi^2 + \frac{\sqrt{q}}{2} q^{ab} \partial_a \phi \partial_b \phi \right], \quad (1)$$

where  $N$  is the lapse function,  $q_{ab}$  is the spatial metric,  $q = \det(q_{ab})$ , and the canonical phase space variables are

$(\phi, P_\phi)$ . Instead of  $(\phi, P_\phi)$ , we use the basic variables

$$\phi_f \equiv \int d^3x \sqrt{q} f(\mathbf{x}) \phi(\mathbf{x}), \quad U_\lambda \equiv \exp\left(\frac{i\lambda P_\phi}{\sqrt{q}}\right), \quad (2)$$

where the smearing function  $f(\mathbf{x})$  is a scalar. Such variables are a typical choice for polymer quantization [8, 10] Since  $P_\phi$  transforms as a density under coordinate transformations, the  $\sqrt{q}$  factor in the second definition is required to make the argument of the exponent a scalar; *this factor turns out to be crucial for obtaining our results*. The parameter  $\lambda$  is a spacetime constant with dimensions of  $(\text{mass})^{-2}$ . These variables satisfy the Poisson algebra

$$\{\phi_f, U_\lambda\} = i f \lambda U_\lambda. \quad (3)$$

From here on we specialize to an FRW spacetime with spatial metric  $q_{ab} = a^2(t)\delta_{ab}$  and choose the proper time gauge  $N = 1$ . Homogeneity requires that the smearing function be constant, so we select  $f(\mathbf{x}) = 1$ . Employing a standard box normalization to regulate the spatial integration in (2) gives the symmetry reduced variables

$$\phi_f = V_0 a^3 \phi, \quad U_\lambda = \exp(i\lambda P_\phi / a^3), \quad (4)$$

where  $V_0 = \int d^3x$  is a fiducial comoving volume. The Poisson bracket of the reduced variables is the same as that of the unreduced ones (3).

Quantization proceeds by realizing the Poisson algebra (3) as a commutator algebra on a suitable Hilbert space; the choice for polymer quantization has the basis  $\{|\lambda\rangle | \lambda \in \mathbb{R}\}$  with inner product

$$\langle \lambda' | \lambda \rangle = \delta_{\lambda, \lambda'}, \quad (5)$$

where  $\delta$  is the generalization of the Kronecker delta to the real numbers [8]. The operators  $\hat{\phi}_f$  and  $\hat{U}_\lambda$  have the action

$$\hat{\phi}_f |\lambda\rangle = \lambda |\lambda\rangle, \quad \hat{U}_{\lambda'} |\lambda\rangle = |\lambda + \lambda'\rangle; \quad (6)$$

i.e.,  $|\lambda\rangle$  is an eigenstate of the smeared field operator  $\hat{\phi}_f$ , and  $\hat{U}_\lambda$  is the generator of field translation.

With this realization it is evident that configuration eigenstates are normalizable. This is one of the main difference between the polymer and Schrödinger quantization schemes. It is because of this that the momentum operator does not exist in this quantization, but must be defined indirectly using the translation generators by the relation

$$P_\phi^\lambda = \frac{a^3}{2i\lambda} (U_\lambda - U_\lambda^\dagger). \quad (7)$$

At this stage it is convenient to fix the polymer quantization scale by setting  $\lambda = \lambda_\star = 1/M_\star^2$  in the momentum operator; we set  $P_\phi^\star \equiv P_\phi^{\lambda_\star}$ . The limit  $M_\star \rightarrow \infty$  gives

$P_\phi^\star \rightarrow P_\phi$  at the classical level, but in polymer quantization  $M_\star$  remains a fixed and finite scale.

Our prescription for cosmological dynamics is through the effective Hamiltonian constraint

$$H_g + \langle \psi | \hat{H}_\phi | \psi \rangle = 0, \quad (8)$$

where  $|\psi\rangle$  is a suitably chosen matter semi-classical state, and  $H_\phi = V_0 (P_\phi^\star)^2 / 2a^3$  in an FRW background. Using (7) gives

$$\langle \hat{H}_\phi \rangle \equiv V_0 a^3 \rho_{\text{eff}}, \quad \rho_{\text{eff}} = \frac{1}{8} M_\star^4 [2 - \langle \hat{U}_{2\lambda_\star} \rangle - \langle \hat{U}_{2\lambda_\star}^\dagger \rangle], \quad (9)$$

where  $\rho_{\text{eff}}$  is the quantum corrected matter density. To compute the expectation values of the translation operator, we take  $|\psi\rangle$  to be a Gaussian coherent state peaked at the phase space values  $(\phi_0, P_\phi)$ . Such a state is

$$|\psi\rangle = \frac{1}{\mathcal{N}} \sum_{k=-\infty}^{\infty} c_k |\lambda_k\rangle, \quad c_k \equiv e^{-(\phi_k - \phi_0)^2 / 2\sigma^2} e^{-iP_\phi \phi_k V_0}, \quad (10)$$

where  $\phi_k = \lambda_k / V_0 a^3$  is an eigenvalue of the scalar field operator derived from  $\hat{\phi}_f$  in Eq. (6). The scalar configuration points  $\lambda_k$  are chosen such that the Gaussian profile is well sampled; the simplest example is a uniform sampling.

The normalization factor is  $\mathcal{N}^2 = \sum_k |c_k|^2 \simeq V_0 a^3 \sigma \sqrt{\pi}$ , where the symbol  $\simeq$  means that the sum is approximated by an integral. This state gives the expectation value

$$\langle \hat{U}_{\lambda_\star} \rangle \simeq e^{i\Theta} e^{-\Theta^2 / 4\Sigma^2}, \quad (11)$$

$$\Theta \equiv \lambda_\star P_\phi a^{-3} = P_\phi M_\star^{-2} a^{-3}, \quad \Sigma \equiv \sigma V_0 P_\phi. \quad (12)$$

Using this result the quantum corrected effective energy density (9) is

$$\rho_{\text{eff}}(a, P_\phi; \sigma, M_\star) \simeq \frac{1}{4} M_\star^4 [1 - e^{-\Theta^2 / \Sigma^2} \cos 2\Theta]. \quad (13)$$

Its dependence on  $M_\star$  is through the momentum operator, and on  $\sigma$  and  $P_\phi$  through the choice of state.

The variables  $\Theta$  and  $\Sigma$ , and hence  $\rho_{\text{eff}}$ , are invariant under re-scalings of the spatial coordinates, or equivalently the re-definition of the scale factor  $a$ :

$$\mathbf{x} \rightarrow \ell \mathbf{x}, \quad a \rightarrow \ell^{-1} a, \quad V_0 \rightarrow \ell^3 V_0, \quad P_\phi \rightarrow \ell^{-3} P_\phi. \quad (14)$$

This physically necessary property of the quantum corrected energy density follows directly from the  $\sqrt{q}$  factor in the definition (2) of  $U_\lambda$ . Furthermore,  $\Theta$  and  $\Sigma$  have natural interpretations:

$$\Theta = \sqrt{2\rho_{\text{cl}}/M_\star^4}, \quad (15)$$

where  $\rho_{\text{cl}} = P_\phi^2 / 2a^6$  is the classical density of a free scalar field in an FRW universe, and

$$\frac{\Delta P_\phi^2}{\langle P_\phi^\star \rangle^2} = \frac{1}{2\Sigma^2} + \mathcal{O}\left(\frac{\Theta^2}{\Sigma^2}\right), \quad (16)$$

in the late time ( $a \rightarrow \infty$ ) limit, where  $\Delta P_\phi^2 = \langle P_\phi^{*2} \rangle - \langle P_\phi^* \rangle^2$  is computed using (7) and (11). From this it is evident that as  $\Delta P_\phi \rightarrow 0$ ,  $\Sigma \rightarrow \infty$  for fixed  $\langle P_\phi^* \rangle$ . Thus  $\Sigma$  measures how “squeezed” the state  $|\psi\rangle$  is in the  $P_\phi$  direction at late times.

**Effective cosmological dynamics** Having derived an expression for the effective density (13), the Friedmann equation, and the evolution equations for  $\phi$  and  $P_\phi$  follow from (8). To derive the latter, the peaking values in the state  $|\psi\rangle$  are treated as a canonically conjugate pair. We find

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho_{\text{eff}}(a, P_\phi; \sigma, M_\star), \quad (17)$$

$$\dot{P}_\phi = 0, \quad \dot{\phi} = \frac{1}{2} M_\star^2 e^{-\Theta^2/\Sigma^2} \sin(2\Theta). \quad (18)$$

The form of  $\rho_{\text{eff}}$  implies that the Hubble parameter is bounded from above  $H^2 \leq 4\pi G M_\star^4/3$ . Hence, the polymer quantization effects ensure that there is no curvature singularity in this model.

The behaviour of  $a(t)$  may be seen from the asymptotics of  $\rho_{\text{eff}}$ . When the scale factor is large (or  $\Theta \lesssim 1$ ) we find

$$\rho_{\text{eff}} \sim \frac{1}{2a^6} \left( P_\phi^2 + \frac{1}{2\sigma^2 V_0^2} \right), \quad a \gtrsim a_{\text{GR}} \equiv \left( \frac{P_\phi}{M_\star^2} \right)^{1/3}. \quad (19)$$

This is coincident with the limit in which polymer quantization reduces to the Schrödinger one; i.e.,  $M_\star \rightarrow \infty$ . In this regime we have  $H \propto a^{-3}$  and  $a \propto t^{1/3}$ , which is the standard result for a massless scalar field coupled to the FRW background in general relativity; hence,  $a_{\text{GR}}$  indicates the beginning of an epoch where the model evolves classically.

Conversely, when the scale factor is small (or  $\Theta \gtrsim \Sigma$ ) we obtain

$$\rho_{\text{eff}} \sim \frac{1}{4} M_\star^4, \quad a \lesssim a_{\text{dS}} \equiv \Sigma^{-1/3} a_{\text{GR}}. \quad (20)$$

That is, the Hubble parameter is constant in the early universe, which gives the asymptotic solution

$$a(t) \propto \exp(H_\star t), \quad H_\star \equiv (2\pi G M_\star^4/3)^{1/2}. \quad (21)$$

This is one of the main results of this work: polymer quantization of a free scalar field coupled to an FRW universe gives an effective de Sitter inflationary phase in the early universe that ends when  $a \sim a_{\text{dS}}$ . During this phase, the Hubble parameter is incredibly close to constant; i.e., when  $a = e^{-N} a_{\text{dS}}$  we find  $|H^2/H_\star^2 - 1| \sim \exp(-\exp 6N)$ . This exponential expansion persists into the past ( $t \rightarrow -\infty$ ), so the number of  $e$ -folds is infinite.

To summarize: The universe undergoes a de Sitter-like expansion when  $a \lesssim a_{\text{dS}}$ , or when the classical energy density is large  $\rho_{\text{cl}} \gtrsim \Sigma^2 M_\star^4$ . When  $a \gtrsim a_{\text{GR}}$ , or  $\rho_{\text{cl}} \lesssim M_\star^4$ , the universe evolves as in GR. In Fig. 1, we illustrate these features of the cosmological dynamics for several specific choices of  $\Sigma$ .

**Emergent universe** The effective early time de Sitter phase is present when the squeezing parameter  $\Sigma$  is finite, but otherwise independent of its actual value. If  $\Sigma \rightarrow \infty$ , the Friedmann equation reduces to  $H^2 = (4\pi G M_\star^4/3) \sin^2 \Theta$ . This has a number of *static* solutions:

$$a(t) = (P_\phi/M_\star^2 n\pi)^{1/3} \equiv a_n, \quad \dot{a}(t) = \ddot{a}(t) = 0, \quad (22)$$

where  $n = 1, 2, 3, \dots$ . Each of these solutions are unstable fixed points representing *Minkowski* 4-geometries. In particular, one can find scale factor solutions with asymptotic behaviour

$$a(t) \sim \begin{cases} c_1 e^{\sqrt{3\pi^3 G} M_\star^2 t} + a_1, & t \rightarrow -\infty \\ c_2 t^{1/3}, & t \rightarrow +\infty \end{cases}, \quad (23)$$

where  $c_1$  and  $c_2$  are constants. This class of solutions reproduces the conventional universe dynamics at late times; however, at early times the universe asymptotes to Minkowski space with  $a \rightarrow a_1$ . We comment that the solutions (23) are reminiscent of the emergent universe scenario proposed by Ellis and Maartens [13] where the initial state was an Einstein-static configuration with positive spatial curvature.

**Discussion** We have explored the cosmological consequences of the idea that quantum translations of the amplitude of a free scalar field are fundamentally discrete. Working with semi-classical quantum states, we found that this simple principle leads to the avoidance of the big bang singularity when such fields are coupled to a homogeneous and isotropic classical universe. Unlike LQC which uses a similar quantization scheme for gravitational degrees of freedom, universes containing polymerized scalar fields approach de Sitter or Minkowski space in the asymptotic past. Thus inflationary or flat universes are past attractors of this model, with classical general relativity recovered at late times.

As can be seen in the effective Friedmann equation (17), our model is characterized by two mass scales. The first of these  $M_\star$  is a fundamental parameter of our quantization procedure, and is directly related the magnitude of discrete field translations. The second parameter  $\sigma$  gives the quantum uncertainty in the field amplitude for the semi-classical states we are considering. As long as  $\sigma$  is finite, we are guaranteed that the universe will undergo inflation at early times. If we assume GUT-scale inflation with  $M_\star \sim 10^{15}$  GeV, we are guaranteed that the maximum density achieved by the scalar field is much less than  $M_{\text{pl}}^4$ , and we may safely neglect the quantum gravity effects predicted by loop quantum cosmology.

Our model has a number of cosmologically attractive features: In contrast to LQC, the de Sitter-like phase is past eternal, so there is sufficient inflation to solve the horizon and entropy problems. (However, this may imply our spacetime is past incomplete [14], which requires further study.) Unlike many other models, there is a

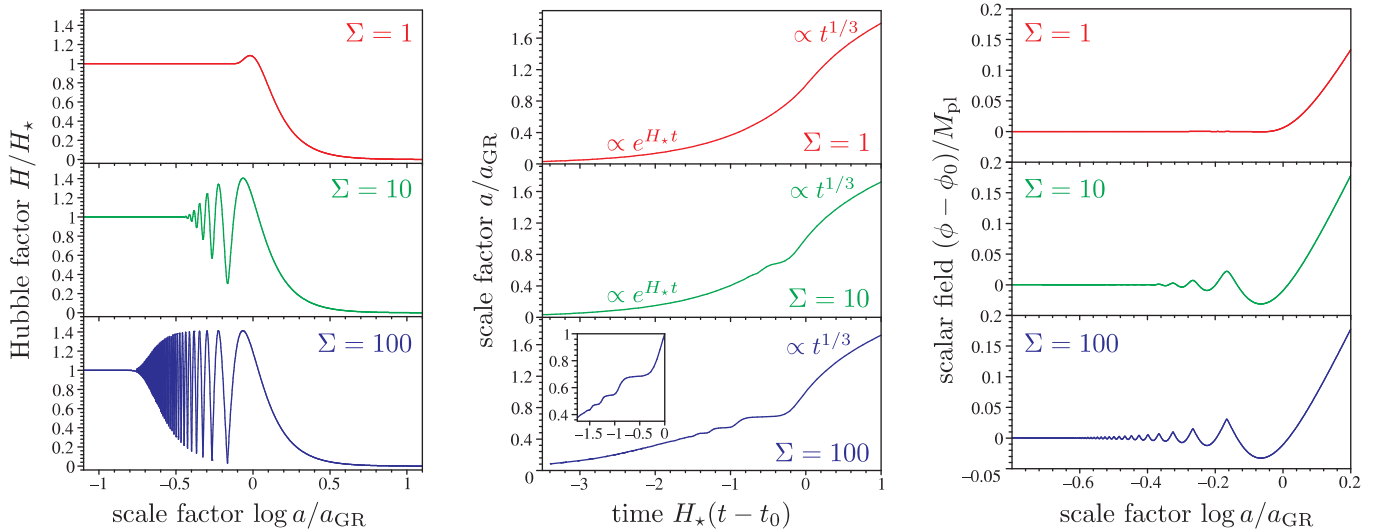


FIG. 1: Behaviour of the Hubble parameter (*left*), scale factor (*center*) and field amplitude (*right*) for various values of  $\Sigma$ . Here,  $\phi_0$  and  $t_0$  are arbitrary constants. We see that  $H$  is virtually constant for  $a \lesssim a_{\text{dS}} = \Sigma^{-1/3} a_{\text{GR}}$ , which leads to an early time de Sitter phase where  $a$  grows exponentially and  $\phi$  is constant. Between  $a_{\text{dS}}$  and  $a_{\text{GR}}$ ,  $H$  and  $\phi$  show oscillatory behaviour while  $a$  evolves in a “stair-step” pattern. When  $a \gtrsim a_{\text{GR}}$ , we recover conventional classical dynamics.

natural end to the inflationary period when the polymer quantization effects become sub-dominant. We do not have to fine-tune parameters to obtain inflation; i.e., just assuming a finite width for the matter quantum state is enough to guarantee the existence of the de Sitter-like phase. Furthermore there exists an interesting transition epoch between the end of the inflationary phase and the onset of the classical period where the scale factor evolves in a pseudo-discrete manner and the scalar field oscillates. The latter may be relevant for reheating the universe via parametric resonance.

An open question in this scenario is the generation of primordial perturbations. The fact that we have nearly de Sitter inflation would suggest that the spectrum of fluctuations produced by this model would compare favourably with observations of the cosmic microwave background, etc. However, some caution is warranted: The polymer quantization procedure will have a significant impact on the behaviour of inhomogeneous  $\phi$  perturbations. An initial study of the dynamics of an inhomogeneous scalar field in the context of polymer quantization can be found in Ref. [15], where the behaviour of  $\phi$  in a Minkowski background is studied. We find that conventional flat space Klein-Gordon equation is replaced by a nonlinear wave equation, and polymer corrections to conventional dynamics depend on both that amplitude and frequency of matter waves. We expect this feature to generalize to the cosmological case. It is worthwhile mentioning that one can derive a lower bound of  $M_* \gtrsim 1$  TeV from applying the flat space results in [15] to the proton-antiproton beam in the Large Hadron Collider.

We are supported by NSERC of Canada and AARMS.

\* Electronic address: ghossain@unb.ca

† Electronic address: vhusain@unb.ca

‡ Electronic address: sseahra@unb.ca

- [1] A. Ashtekar, *Nuovo Cim.* **122B**, 135 (2007), gr-qc/0702030.
- [2] M. Bojowald, *Living Rev. Rel.* **11**, 4 (2008).
- [3] M. Bojowald, *Phys. Rev. Lett.* **86**, 5227 (2001), gr-qc/0102069.
- [4] G. Date and G. M. Hossain, *Phys. Rev. Lett.* **94**, 011302 (2005), gr-qc/0407074.
- [5] A. Ashtekar, T. Pawłowski, and P. Singh, *Phys. Rev. D* **74**, 084003 (2006), gr-qc/0607039.
- [6] M. Bojowald, *Phys. Rev. Lett.* **89**, 261301 (2002), gr-qc/0206054.
- [7] G. Date and G. M. Hossain, *Phys. Rev. Lett.* **94**, 011301 (2005), gr-qc/0407069.
- [8] A. Ashtekar, S. Fairhurst, and J. L. Willis, *Class. Quant. Grav.* **20**, 1031 (2003), gr-qc/0207106.
- [9] H. Halvorson, *Stud. Hist. Phil. Mod. Phys.* **35**, 45 (2004), quant-ph/0110102.
- [10] V. Husain and O. Winkler, *Phys. Rev. D* **69**, 084016 (2004), gr-qc/0312094.
- [11] V. Husain and O. Winkler, *Class. Quant. Grav.* **22**, L127 (2005), gr-qc/0410125.
- [12] V. Husain and O. Winkler, *Class. Quant. Grav.* **22**, L135 (2005), gr-qc/0412039.
- [13] G. F. R. Ellis and R. Maartens, *Class. Quant. Grav.* **21**, 223 (2004), gr-qc/0211082.
- [14] A. Borde, A. H. Guth, and A. Vilenkin, *Phys. Rev. Lett.* **90**, 151301 (2003), gr-qc/0110012.
- [15] G. M. Hossain, V. Husain, and S. S. Seahra, *Phys. Rev. D* **80**, 044018 (2009), 0906.4046.